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Takagi Sugeno Fuzzy Model Based Controller Design of Cart-Ball System

Budiman Azzali Basir
Faculty of Engineering & Technology
Multimedia University
Ayer Keroh, Melaka, Malaysia

Mohammad Noh Ahmad and Abdul Rashid Hussain
Faculty of Electrical Engineering
University Technology Malaysia
Skudai, Johor DT, Malaysia

Abstract— **This paper presents the results of a fuzzy model based controller design via PDC type Takagi-Sugeno (T-S) of cart-ball system. The control aim is to balance the ball at arc center position at any given position of the cart. A fuzzy model based controller is designed based on T-S fuzzy controller via parallel distribution control (PDC) to achieve the control objective. The nonlinear mathematical model for the cart-ball system in this paper is represents by T-S fuzzy models and using PDC type controller as the control law. Based on fuzzy Lyapunov function approach, the stability condition of open loop fuzzy system and the stabilization of closed loop fuzzy system are obtained by solving a set linear matrix inequalities (LMIs). The effectiveness and the feasibility of the PDC type Takagi-Sugeno (T-S) fuzzy control is demonstrated and studied using MATLAB/Simulink software.**

Keywords*— *Cart-Ball System, T-S Fuzzy Model, PDC, LMI.

I. Introduction

A cart-ball system is well used as benchmark of control theory problems owing to the highly nonlinearities and multivariable that consist in the system. The main objective control of this workbench problem is maintained the ball at arc center position at any given position of the cart. Over the year various approaches have been tried [1]-[3]. The purpose of this paper is to presents a PDC type Takagi-Sugeno (T-S) fuzzy control to achieve the control objective of the proposed system which guarantees the stability of the closed loop system.

In recent years, since Takagi and Sugeno have proposed a fuzzy model as a representation of model consisting linear time invariant (LTI) and nonlinear functions [4], the used of Takagi-Sugeno (T-S) fuzzy model has been widely used in many applications [5]-[14]. This is due to its efficiency to control highly and complex nonlinearity in the system. Utilizing the state feedback approach, the same principles is applied to parallel distributed compensation (PDC) which then interpolated with the feedback gains in each of the determined Takagi-Sugeno (T-S) fuzzy rules. Furthermore the global linearized fuzzy model is made up from set of local linearized models which are derived from set of membership functions.

This paper is organized as follows. Section II presents the structured Takagi-Sugeno (T-S) fuzzy approached and PDC derivation in terms of LMI [15]-[16]. Section III presents the system mathematical modeling and with the proposed Takagi-Sugeno (T-S) fuzzy model. Section IV presents the simulation results and analysis. Finally the conclusion is briefed in the last section.

II. Takagi- Sugeno Fuzzy Model

A. *Representation T-S Fuzzy Model*

Model Rule i:

IF $z_i(t)$ is M_i^1 and … and $z_p(t)$ is M_p^i
THEN

$$\dot{x}(t) = A_i x(t) + B_i u(t) \quad (1)$$

where $i = 1,2 \dots, r$. r is the number of IF-THEN rules and $x(t) \in R^n$ is the state vector of the systems. $u(t) \in R^m$ and $y(t) \in R^q$ is the input vector and the output vector of the systems respectively. $B_i \in R^{n\times m}$ and $A_i \in R^{n\times n}$ are the system input matrices and the systems matrices and respectively. $M_j^i, j = 1,2 \dots, p$ is denoted as the j^{ith} fuzzy set for the i^{ith} rule and $z_1(t),..,z_p(t)$ are the known variables functions of state variables. $w_j^i(z_j)$ is denoted as the membership function for j^{ith} fuzzy set M_j^i of i^{ith} and $w_i(z(t) = \prod_{j=1}^{p} M_j^i\,(z(t))\ i = 1,..,r$

Let's denotes the pair of $x(t), u(t),$ as the fuzzy systems output represented as [15]:

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} w_i\,(t)\big((A_i x(t)) + B_i u(t)\big)}{\sum_{i=1}^{r} w_i\,(t)} \quad (2)$$

$z(t)$ is the premise vector for $z(t) = z_1, z_2 \cdots z_p$

As the $z(t)$ is regards as the combination of linear and state vector, thus the weight function can be represented as:

$$h_i\big(z(t)\big) = \frac{w_i(z(t)}{\sum_{i=1}^{r} w_i\,(t)} \quad , i = 1, \cdots, r \quad (3)$$

for all t.

$w_j^i\big(z_j\big)$ denoted as the membership grade for $z_j(t)$ in M_j^i from (2), noting

$$\begin{cases} \sum_{i=1}^{r} h_i\big(z(t)\big) = 1 \\ h_i\big(z(t)\big) \ge 0 \end{cases} \quad i = 1, \cdots, r$$

Therefore the Takagi Sugeno fuzzy model is represents as:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\,(t)\ \big((A_i\, x(t)) + B_i\, u(t)\big) \quad (4)$$

and the output of Takagi Sugeno fuzzy model is represents as:

$$y(t) = \sum_{i=1}^{r} h_i\,(t)\;C_i\,x(t) \qquad (5)$$

B. Fuzzy Controller Design via PDC

Controller design for fuzzy model using PDC approach that derived the controller form the fuzzy model itself as both the controller rules and the plant rules consists of the same weights if fuzzy set [4][15]. Therefore the PDC controller rules are as the following:

Controller rule i:

IF $z_i(t)$ is M_i^1 and … and $z_p(t)$ is M_p^i

THEN $u(t) = -F_i$ for $i = 1, \cdots, r$

Thus overall fuzzy controller rules with the consequent parts are the actual linear controller can be represents as:

$$u(t) = -\sum_{i=1}^{r} h_i\,(t)\;F_i\,x(t) \qquad (6)$$

Controller design for fuzzy model using PDC approach is by finding the feedback gains $F_i \in \Re^{m\times n}$ for each local feedback in the consequent parts. Substituting (6) into (3), we can represent the closed loop model of the system as:

$$x(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)h_j\left(z(t)\right)\left(A_i - B_i\,F_j\right)x(t) \qquad (7)$$

The stability analysis of the T-S fuzzy control system can be determined by using a Lyapunov function candidate stated as [12]:

$$V\big(x(t)\big) = x^T P x.$$

Where $P = P^T \in R^{n\times n} > 0$ such that $\dot{V}\big(x(t)\big) < 0$ in all nonzero trajectory. Thus the system is quadratically stable for all *t*.

Theorem 1: [4] The fuzzy model in (2) is said to be asymptotically stable when there exist a common positive definite matrix *P* which satisfies the following Lyapunov inequalities:

$$G_{ii}^T P + P G_{ii}^T < 0 \qquad (8)$$

$$\left(\frac{G_{ij}+G_{ji}}{2}\right)^T P + P\left(\frac{G_{ij}+G_{ji}}{2}\right) < 0 \quad \text{for } i < j \qquad (9)$$

where $G_{ij} = A_i - B_i F_j$

For each $i, j \in \{1, 2 \cdots r\}$ except for pairs of $\forall t\; h_i\big(z(t)\big)h_j\big(z(t)\big) \neq 0$ where r is the number of if-then rules.

Using linear matrix inequalities (LMI's) transformations [18], yields the exponential stability condition [15]. Then transforms into LMIs term by multiplication of P^{-1}. Denoting $X = P^{-1}$ and $M_i = F_i X$ yields:

$$X\,A_i^T + A_i X - B_i M_i - M_i^T B_i^T < 0 \quad i = 1,\ldots,r \qquad (10)$$

$$X\left(A_i^T + A_j^T\right) + \left(A_i + A_j\right)X - \left(B_i M_j + B_j M_i\right) - \left(B_i M_j + B_j M_i +\right)^T < 0$$
$$i < j \le r \qquad (11)$$

III. System Modelling

A. Mathematical Model

The cart-ball system diagram is shown in Figure. 1. Assuming that the direction towards the right is considered to be positive while left vice versa and frictions forces are neglected.

Forces in x-direction balances gives the mass times of the cart acceleration at mass point. Therefore it will equal with the external force of the system. Thus the cart-ball mechanical system is as follows:

$$M\frac{d^2x}{dt^2} + m\frac{d^2}{dt^2}\left[y + (R+r)\sin\varphi\right] = F \qquad (12)$$

Expanding (12) we get,

$$(M+m)\ddot{y} - (M+m)(\sin\varphi)\dot{\varphi}^2 + m(R+r)(\cos\varphi)\ddot{\varphi} = F \qquad (13)$$

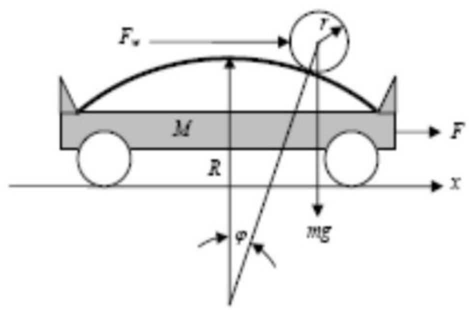

Figure 1. Free body diagram of system

The force components balanced by torque can be written as:

$$F_x\cos\varphi\,(R+r) - \left(F_y\sin\dot{\varphi}\right) = mg\sin\varphi\,(R+r) \qquad (14)$$

Rearranging (13)

$$m(R+r)\ddot{\varphi} = mg\sin\varphi - m\ddot{y}\cos\varphi \qquad (15)$$

By substituting (15) in (12) gives:

$$(M+m)\ddot{y} - m(R+r)\sin\varphi\,\dot{\varphi}^2 + mg\cos\varphi\sin\varphi - m\ddot{y}\cos^2\varphi = F \qquad (16)$$

$$\left(m(R+r)\cos^2\varphi - (M+m)\,(R+r)\,\ddot{\varphi}\right) = F\cos\varphi$$
$$(M+m)g\sin\varphi + m(R+r)\cos\varphi\dot{\varphi}^2 \qquad (17)$$

Rearranging (16) and (17) gives:

$$\ddot{y} = \frac{F + m(R+r)\sin\varphi\dot{\varphi}^2 - mg\cos\varphi\sin\varphi}{M+m-m\cos^2\varphi} \qquad (18)$$

$$\ddot{\varphi} = \frac{F\cos\varphi - (M+m)g\sin\varphi + m(R+r)(\cos\varphi\sin\varphi)\dot{\varphi}}{m(R+r)\cos^2\varphi - (M+m)(R+r)} \qquad (19)$$

Defining the state vector *x* as

$$x_1 = \varphi, x_2 = \dot{\varphi}, x_3 = y, x_4 = \dot{y} \qquad (20)$$

The, the nonlinear equation can be written in state space as:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \frac{Fcosx_1 - (M+m)gsinx_1 + m(R+r)(cosx_1 sinx_1)x_2^2}{m(R+r)cos^2x_1 - (M+m)(R+r)}$$

$$\dot{x_3} = x_4$$

$$\dot{x_4} = \frac{F + m(R+r)(sinx_1)x_2^2 - mgcosx_1 sinx_1}{M+m-mcos^2x_1} \qquad (21)$$

B. TS Fuzzy Model of Cart-Ball System

The fuzzy model via Takagi Sugeno approach for this system is based on local approximation approach to approximate nonlinear terms. The plant rules and the controller rules are blended to make up the overall control system for T-S fuzzy control.

The main objective control of this workbench problem is maintained the ball at arc center position at any given position of the cart. This can be achieved by selecting the approximate range for $x_1(t) \in (-\pi/2, \pi/2)$. Therefore the cart-ball system can be approximate by following 2-rule model.

When $x_1(t)$ is near 0, the system can represent in state space form as:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \frac{F - (M+m)gx_1 + m(R+r)x_1x_2^2}{m(R+r) - (M+m)(R+r)}$$

$$\dot{x_3} = x_4$$

$$\dot{x_4} = \frac{F + m(R+r)x_1x_2^2 - mgx_1}{M} \qquad (22)$$

When $x_1(t)$ is near ±π/2, the system can represent in state space form as:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \frac{F\beta - 2g(M+m)/\pi + 2\beta m(R+r)x_2^2/\pi}{m\beta^2(R+r) - (M+m)(R+r)}$$

$$\dot{x_3} = x_4$$

$$\dot{x_4} = \frac{F + 2m(R+r)x_2^2/\pi - 2\beta mg/\pi}{M+m-m\beta^2} \qquad (23)$$

Where $\beta = cos(88^0)$. Therefore the model rules are as follows:

Rule 1: IF $x_1(t)$ is about 0,
THEN $\dot{x}(t) = A_1x(t) + B_1u(t)$
Rule 2: IF $x_1(t)$ is about ±π/2
THEN $\dot{x}(t) = A_2x(t) + B_2u(t)$

Thus PDC controller can be stated as:

$$u(t) = -\sum_{i=1}^{2} h_i\left(z(t)\right)F_i x(t) \qquad (24)$$

IV. SIMULATION RESULT

In this section, the simulation result of Takagi Sugeno fuzzy modeling with PDC controller of cart ball system is presented.

Table 1: Cart-ball system physical values

Parameter	Symbol	Value
Arc radius	R	0.5 m
Weight of the cart	M	1 kg
Position of the cart	y	
Force	F	
Ball radius	r	0.025 m
Angular deviation	$\varphi = \theta$	max 0.6rad
Weight of the ball	m	0.675 kg
Gravity	g	9.81 ms^{-2}

Given the physical data from Table I, the state space matrices for T-S fuzzy model are obtained as follows:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 31.30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -6.62 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 29.47 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -4.21 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ -1.91 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -2.82 \\ 0 \\ 0.99 \end{bmatrix}$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D_1 = D_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

The fuzzy PDC controller gains are obtained by solving set of LMIs (8)-(11) thus the resulting positive definite P are as follows:

$$P = \begin{bmatrix} 0.0932 & 0.0199 & 0.0094 & 0.0174 \\ 0.0199 & 0.0065 & 0.0023 & 0.0056 \\ 0.0094 & 0.0023 & 0.0023 & 0.0023 \\ 0.0174 & 0.0056 & 0.0023 & 0.0064 \end{bmatrix}$$

and

$$F_1 = [-44.0660 \quad -5.7792 \quad -2.4581 \quad -4.5469]$$

$$F_2 = [-32.9789 \quad -4.6256 \quad -1.9510 \quad -3.7025]$$

In this simulation, $x(t)$ denotes the for state variables and assuming the initial condition for $x(0)$ are $[0.3 \quad 0.3]^T$. The simulation results for $x(t)$ shown in Figure 2. The asymptotic stability of the system is determined by the set of LMI's (9) and (10).

Figure 3 and Figure 4 show the output of the system where the cart required to moves to center position and maintaining the ball at the top of cart. The results show that feasibility and the effectiveness of the control approach. Thus shows that the design approach is significantly better than the classical approach for cart ball system [3]. Furthermore the asymptotically stability can be determined by the feasibility of the LMI's.

There are also others approach using relaxed stabilization conditions and nonquardratic Lyapunov functions to obtain conditions for stability for T-S fuzzy model control [16][17].

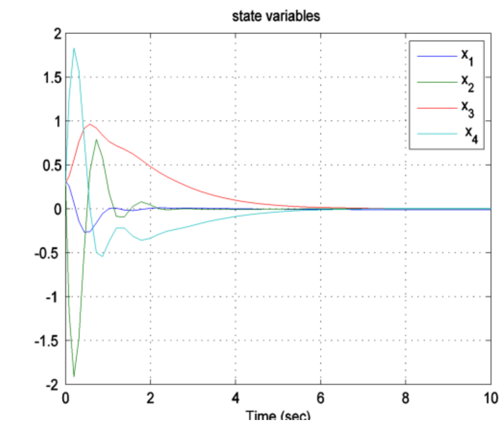

Figure 2. State variables of Cart-Ball system

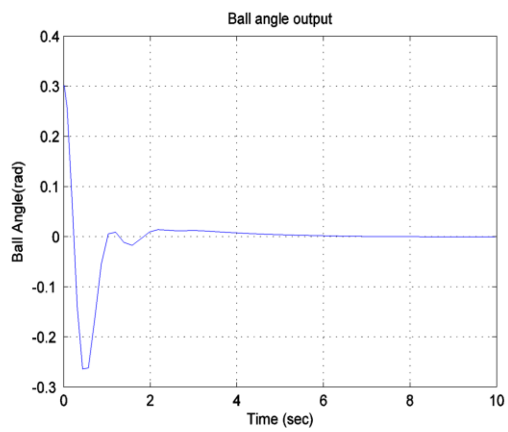

Figure 3. Ball angle output

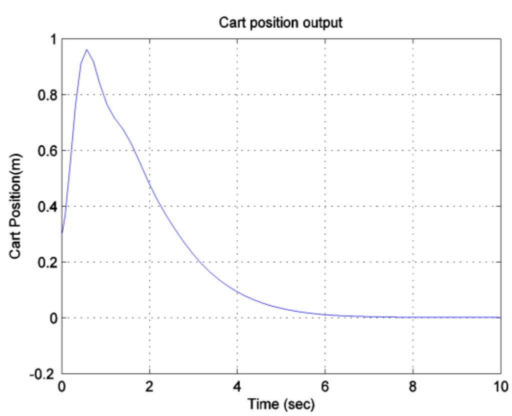

Figure 4. Ball angle output

V. Conclusion

This paper a presents Takagi Sugeno fuzzy modeling and controller design of cart ball system using PDC control law.

It has been shown that the approach design is conceptually simple and natural even to highly complex nonlinear system. Furthermore, the control design problem and the analysis of stability are formulated Linear Matrix Inequality (LMI) formulations. The proposed controller is simulated using Matlab/Simulink (R2009a).

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