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A New Note on Summability of Factored Infinite Series

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***Abstract*—In this paper, two main theorem dealing with infinite and factored Fourier series, which generalizes some known results, has been generalized to $|A, p_n; \delta|_k$ summability method. This new theorem also includes several known and new results.**

Keywords— ***Summability factors, absolute matrix summability, Fourier series, infinite series, Hölder inequality, Minkowski inequality.***

I. Introduction

Definition 1. Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty,\ n \to \infty,\ (P_{-i} = p_{-i} = 0,\ i \geq 1). \quad (1)$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (2)$$

defines the sequence (t_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [11]).

Definition 2. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$ if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\Delta \sigma_{n-1}|^k < \infty. \quad (3)$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C,1|_k$ (resp. $|\bar{N}, p_n|$) summability (see [10]).

Definition 3. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$ if (see [7])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |\Delta \sigma_{n-1}|^k < \infty. \quad (4)$$

where $\Delta \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v,\ n \geq 1.$

In the special case, when $p_n = 1$ for all values of n (resp. $\delta = 0$), $|\bar{N}, p_n; \delta|_k$ summability is the same as $|C,1; \delta|_k$ (resp. $|\bar{N}, p_n|_k$) summability.

II. Known Results

Many works dealing with some absolute summability methods of infinite and Fourier series have been done (see [1-2], [4-8], [12-15],[21]). Among them, in [16] Özarslan has proved the following theorem.

Theorem 1. Let $k \geq 1$. If the sequence (s_n) is bounded and the sequences (λ_n) and (p_n) satisfy the following conditions

$$\sum_{n=1}^{m} p_n |\lambda_n| = O(1) \quad as \quad n \to \infty, \quad (5)$$

$$\sum_{n=1}^{m} p_n |\Delta \lambda_n| = O(1) \quad as \quad n \to \infty, \quad (6)$$

$$p_{n+1} = O(p_n), \quad (7)$$

then the series $\sum a_n \lambda_n P_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

An Application of Absolute Matrix Summability to Fourier Series

Let $f(x)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$. The Fourier series of $f(x)$ is

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$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} C_n(x) \qquad (8)$$

where (a_n) and (b_n) denote the Fourier coefficients.

The convergence of Fourier series can be ensured by local hypothesis, that is to say, the behavior of the convergence of Fourier series for a particular value of x depends on the behavior of the function in the immediate neighbourhood of this point only (see [20]).

Theorem 2. ([16]) Let $k \geq 1$.The summability $\left|\overline{N}, p_n\right|_k$ of the series $\sum C_n(x)\lambda_n P_n$ at a point is a local property of a generating function if the conditions (5) and (6) are satisfied.

Definition 4. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0,1,... \qquad (9)$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$if (see [19])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty \qquad (10)$$

and it is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [18])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty \qquad (11)$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \qquad (12)$$

If we take $p_n = 1$ for all values of n, $|A, p_n|_k$ summability is the same as $|A|_k$ summability. Also, if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n|_k$ summability is the same as $\left|\overline{N}, p_n\right|_k$ summability.

and also it is said to be summable $|A, p_n;\delta|_k$, $k \geq 1$, and $\delta \geq 0$ if (see [17])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left|\bar{\Delta} A_n(s)\right|^k < \infty. \qquad (13)$$

III. Main Results

The aim of this paper is to prove a more general theorem which includes some of the above-mentioned result as a special cases.

Theorem 3. Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let (s_n) be a bounded sequence and suppose that $A = (a_{nv})$ is a positive normal matrix such that

$$\bar{a}_{n0} = 1, \quad n = 0,1,..., \qquad (14)$$

$$a_{n-1,v} \geq a_{nv} \text{ for } 1 \leq v \leq n-1, \qquad (15)$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \qquad (16)$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v(\hat{a}_{nv})| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k - 1}\right\} \quad as \quad m \to \infty, \qquad (17)$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\right\} \quad as \quad m \to \infty. \qquad (18)$$

If a sequence (λ_n) and (p_n) holds the following conditions,

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k} p_n |\lambda_n| = O(1) \quad as \quad m \to \infty, \qquad (19)$$

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k} p_n |\Delta\lambda_n| = O(1) \; as \quad m \to \infty, \qquad (20)$$

$$p_{n+1} = O(p_n), \qquad (21)$$

then the series $\sum a_n \lambda_n P_n$ is summable $|A, p_n;\delta|_k$.

Theorem 4. Let $k \geq 1$ and $0 \leq \delta < 1/k$. The summability $|A, p_n;\delta|_k$ of the series $\sum C_n(x)\lambda_n P_n$ at a point is a local property of a generating function if all the conditions of Theorem 3 are satisfied.

We need the following lemma for the proof of our theorem.

Lemma 5 (see [16]) If the sequences (λ_n) and (p_n) satisfy the conditions (5) and (6) of Theorem 1, then $P_m |\lambda_m| = O(1)$ as $m \to \infty$.

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Proof of Theorem 3

(I_n) denotes the A-transform of the series $\sum a_n \lambda_n P_n$. Then, we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v P_v.$$

Applying Abel's transformation to this sum, we get that

$$\bar{\Delta} I_n(x) = \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv} \lambda_v) P_v \sum_{r=1}^{v} a_r + \hat{a}_{nn} \lambda_n P_n \sum_{v=1}^{n} a_v$$
$$= \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv} \lambda_v P_v) + a_{nn} \lambda_n P_n s_n$$
$$= \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv}) \lambda_v P_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta \lambda_v P_v s_v$$
$$- \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} p_{v+1} s_v + a_{nn} \lambda_n P_n s_n$$
$$= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.$$

To complete the proof of Theorem 3, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n / p_n)^{\delta k + k - 1} \left| I_{n,r}(x) \right|^k < \infty, \text{ for } r = 1,2,3,4.$$

Proof of Theorem 4

The convergence of the Fourier series at $t = x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 4 is a consequence of Theorem 3 and Lemma 5 (see [9]).

IV. Conclusions

Corollary 1. If we take $\delta = 0$ in Theorem 3, then we obtain Theorem 1 dealing with $\left| A, p_n \right|_k$ summability.

Corollary 2. If we take $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3, then we obtain a new theorem concerning with $\left| \bar{N}, p_n; \delta \right|_k$ summability factors of Fourier series.

Corollary 3. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 3, then we get a result concerning $\left| C, 1; \delta \right|_k$ summability factors of Fourier series (see [10]).

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References

1. S. Baron, "Local property of Fourier series", Tartu Riikl. Ül. Toimetised, Vih., **1965**, 177, 106-120.
2. S. N. Bhatt, "An aspect of local property of summability of the factored Fourier series", Proc. Natl. Inst. Sci. India, **1960**, 26, 69-73.
3. H. Bor, "On two summability methods", Math. Proc. Cambridge Philos Soc., **1985**, 97, 147-149.
4. H. Bor," On the absolute Riesz summability factors", Rocky Mountain J. Math., **1994**, 24, 1263-1271.
5. H. Bor, "Local property of summability of factored Fourier series", Bull. Inst. Math. Acad. Sinica,**1989**, 17, 165-170.
6. H. Bor, "On the local property of summability of factored Fourier series", J. Math. Anal. Appl.,**1992**, 163, 220-226.
7. H. Bor, "On the local property of $|\bar{N}, p_n; \delta|_k$ summability of factored Fourier series", J. Math. Anal. Appl., **1993**, 179, 646-649.
8. H. Bor, "Local properties of Fourier series", Int. J. Math. Sci., **2000**, 23, 703-709.
9. K. K. Chen, "Functions of bounded variation and the Cesaro means of Fourier series", Acad. Sin. Sci.,**1945**, Record 1, 283-289,
10. T. M. Flett, " On an extension of absolute summability and some theorems of Littlewood and Paley", Proc. London Math. Soc., **1957**, 7, 113-141.
11. G. H. Hardy, "Divergent Series", Oxford Univ. Press, Oxford, **1949**.
12. M. Izumi, S. Izumi, "Localization problem of the absolute Riesz and absolute Nörlund summabilities of Fourier series", Canad. J. Math., **1970**, 22, 615-625.
13. K. Matsumoto, "Local property of the summability", Tôhoku Math. J., **1956**, 2, 114-124.
14. S. M. Mazhar, "On the summability factors of infinite series", Publ. Math. Debrecen, **1966**, 13,229-236.
15. K. N. Mishra, "Multipliers for summability of Fourier series", Bull. Inst. Math. Acad. Sinica,**1986**, 14, 431-438.
16. H. S. Özarslan, "A note on $\left| \bar{N}, p_n \right|_k$ summability factors", Int. J. Pure Appl. Math.,**2004**, 13, 485-490.

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17. H. S. Özarslan, H. N. Öğdük, "Generalizations of two theorems on absolute summability methods", Aust. J. Math. Anal. Appl., **2004**, 1(2), Art. 13,7.
18. W. T. Sulaiman, "Inclusion theorems for absolute matrix summability methods of an infinite series (IV)", Indian J. Pure and Appl. Math., **2003**, 34 (11), 547-1557.
19. N. Tanovic-Miller, "On strong summability", Glasnik Matematicki, **1979**, 34 (14), 87-97.
20. E. C. Titchmarsh, " The Theory of Functions", Oxford University Press London, **1961**.
21. Ş. Yıldız, "A new theorem on local properties of factored Fourier series", Bull Math. Anal. Appl., **2016**, 8 (2), 1-8.